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THEORY OF STABILITY OF THIN ELASTIC HETEROGENEOUS ANISOTROPIC PLATES OF VARIABLE RIGIDITY

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THEORY OF STABILITY OF THIN ELASTIC HETEROGENEOUS ANISOTROPIC PLATES OF VARIABLE RIGIDITY

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ABSTRACT. This article presents an expression for the total potential energy, the differential equation of the elastic surface of a plate, and the boundary conditions for anisotropic heterogeneous plates with variable ridigity, assuming that the middle surface of the plate is simultaneously the surface of both elastic and geometric symmetry. It is shown that Kirchhoff's boundary conditions for the bending of plates cannot always be extended to the buckling of plates.

An ever increasing number of scientific works are presently being devoted to the question of studying the strain and stress of both homogeneous and heterogeneous anisotropic bodies. /139*

S. G. Mikhlin [1], S. G. Lekhnitskiy [2] and H. M. Savin [3] have accomplished significant work in this direction. These authors were the first to formulate and solve several extremely important fundamental questions regarding the plane theory of elasticity of anisotropic bodies.

Some particular cases of anisotropy were considered by Saint-Venant [4], Voigt [5], Somigliana [6], M. Huber [7], Ya. I. Sekerzh-Zen'kovich [8], L. I. Balabukh [9], G. G. Rostovtsev [10], S. V. Serensen [11], and others.

Saint-Venant, Voigt, and Somigliana were concerned with the study of the state of stress of an anisotropic body having the shape of a long cylinder.

M. T. Huber examined the lateral bending of orthotropic plates.

Ya. I. Sekerzh-Zen'kovich and L. I. Balabukh studied the buckling of plywood plates.

G. G. Rostovtsev devoted his article to the question of the reduced width of an orthotropic plate.

S. V. Serensen, in his course on the theory of elasticity, derives a fundamental equation of the plane problem for an orthotropic medium and gives the solution for a case of the bending of plane beams, and also discusses the works of several other authors devoted to questions concerning the theory of elasticity of anisotropic bodies.

This article considers the question of the stability of anisotropic

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heterogeneous thin plates of variable rigidity with a middle surface that is simultaneously the plane of elastic and geometric symmetry.

The article gives the derivation of a Brian-Timoshenko equation, an expression for the total potential energy, and also a differential equation for the elastic surface of a plate and boundary conditions.

As far as we know, neither the Brian-Timoshenko equation, the expression of total potential energy, nor the differential equation of the elastic surface of an anisotropic plate of variable rigidity are encountered in the literature, not to speak of the boundary conditions, which have not been derived for the case of buckling even for an isotropic homogeneous plate of constant rigidity. The boundary conditions derived by Kirchhoff [12] for the particular case of loading of a plate that is under conditions of lateral bending have always been automatically extended to the case of buckling. As we will see below, these boundary conditions cannot always be applied to the case of buckling. /140

1. Basic Premises and Hypotheses

Let us imagine a plate whose lateral surface is formed by a cylinder $f(x,y) = C$, while its upper and lower surfaces are represented by the equation

$$z_k = (-1)^k h_k(x, y) \qquad (k = 1, 2). \tag{1}$$

Let us further assume that the plate is thin and has surfaces of elastic and geometric symmetry.

We shall take the middle surface of the plate as the plane xy.

The condition of geometric symmetry has the following form:

$$h_1(x, y) = h_2(x, y). \tag{2}$$

The condition of elastic symmetry in the case under consideration will be [13]:

$$a_{14} = a_{24} = a_{15} = a_{25} = a_{34} = a_{35} = a_{46} = a_{56} = 0, \tag{3}$$

where a_{ik} (x,y) are the coeffecients of elasticity of the given plate material.

Let us assume that the known hypotheses [13, 14, 15] that are used when deriving the equations of equilibrium of thin isotropic homogeneous plates of constant rigidity can be extended also to thin anisotropic heterogeneous plates of variable rigidity, the middle surface of which is simultaneously the surface of elastic and geometric symmetry (under the condition that the upper and lower surfaces of the plate are smooth surfaces).

2. Brian-Timoshenko Equation

Let us consider a plate that is in a plane state of stress under the action of forces parallel to its plane.

If the forces acting on the plate are essentially compressive, the plate may be in one of two states of stress: plane, when it has not yet been deflected, but is being deformed in its plane; and deflected, when, in addition to deformation in its plane, it is also being bent.

The work of the external forces in the first state, which is equal to the work of the internal stresses, can be represented in the following form [15, 16, 17]:

$$A_1 = \iint \left[T_1 \frac{\partial u}{\partial x} + T_2 \frac{\partial v}{\partial y} + S \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dx\, dy, \tag{4}$$

Where A_1 is the work of the external forces in the first state of the plate; u and v are displacements of points of the middle surface of the plate in the direction of the x and y axes; T_1 and T_2 are the forces applied to a unit length and extending the plate in the direction of the x and y axes; and S is a tangential force, which is also applied to a unit length and lies in the plane of the plate.

The work of the internal forces in the second state of the plate will consist of two parts [15, 16]: the work of the longitudinal forces with plate distortion taken into account /141

$$\iint \left\{ T_1 \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + T_2 \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \right] + S \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right] \right\} dx\, dy \tag{5}$$

and the work of plate bending

$$-\frac{1}{2} \iint \left\{ M_1 \frac{\partial^2 w}{\partial x^2} + M_2 \frac{\partial^2 w}{\partial y^2} + 2H \frac{\partial^2 w}{\partial x \partial y} \right\} dx\, dy, \tag{6}$$

where M_1, M_2 and H are the elastic moments applied to a unit length of the plate, and which have the following form for an anisotropic heterogeneous plate of variable rigidity [18]:

$$M_1 = -J(x, y) \left[\alpha_{11} \frac{\partial^2 w}{\partial x^2} + \alpha_{12} \frac{\partial^2 w}{\partial y^2} + 2\alpha_{16} \frac{\partial^2 w}{\partial x \partial y} \right],$$

$$M_2 = -J(x, y) \left[\alpha_{21} \frac{\partial^2 w}{\partial x^2} + \alpha_{22} \frac{\partial^2 w}{\partial y^2} + 2\alpha_{26} \frac{\partial^2 w}{\partial x \partial y} \right],$$

$$H = -J(x,y)\left[\alpha_{61}\frac{\partial^2 w}{\partial x^2} + \alpha_{62}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right], \tag{7}$$

$$J(x,y) = \int_{-h}^{h} z^2\,dx,$$

where α_{ik} (x,y) are the coefficients of elasticity of the given material, and represented in terms of the coefficients given above by means of the formula [18]

$$\alpha_{ik}(x,y) = \frac{a_{ik}a_{33} - a_{i3}a_{3k}}{a_{33}}. \tag{8}$$

The coefficients α_{ik} (x,y), in turn, can be represented in terms of the following "technical" coefficients, which are more familiar to us:

$$\alpha_{11} = \frac{1}{K}\left(\frac{1}{E_2G} - \beta_2^2\right), \quad \alpha_{12} = \frac{1}{K}\left(\frac{\nu_2}{E_2G} + \beta_1\beta_2\right), \quad \alpha_{16} = \frac{1}{K}\left(\frac{\beta_1}{E_2} + \frac{\nu_2\beta_2}{E_1}\right),$$

$$\alpha_{21} = \frac{1}{K}\left(\frac{\nu_1}{E_1G} + \beta_1\beta_2\right), \quad \alpha_{22} = \frac{1}{K}\left(\frac{1}{E_1G} - \beta_1^2\right), \quad \alpha_{26} = \frac{1}{K}\left(\frac{\beta_2}{E_1} + \frac{\nu_1\beta_1}{E_2}\right),$$

$$\alpha_{61} = \frac{1}{K}\left(\frac{\beta_1}{E_2} + \frac{\nu_2\beta_2}{E_1}\right), \quad \alpha_{62} = \frac{1}{K}\left(\frac{\beta_2}{E_1} + \frac{\nu_1\beta_1}{E_2}\right), \quad \alpha_{66} = \frac{1}{K}\left(\frac{1}{E_1E_2} - \frac{\nu_1\nu_2}{E_1E_2}\right),$$

where

$$K = \frac{1}{E_1E_2G}(1 - \nu_1\nu_2) - \beta_1\beta_2\left(\frac{\nu_1}{E_1} + \frac{\nu_2}{E_2}\right) + \left(\frac{\beta_1^2}{E_2} - \frac{\beta_2^2}{E_1}\right),$$

E_1 and E_2 are Young's moduli for extension (compression) in the direction of the x and y axes; ν_1 and ν_2 are Poisson's ratios which characterize transverse compression of the material upon extension along the x and y axes; G is the shear modulus for planes parallel to the xy plane; β_1 and β_2 are coefficients which /142 characterize elongation due to the effect of tangential stress X_y, or, expressed in different terms, shear due to normal stresses X_x and Y_y.

It follows from the relationship $\alpha_{ik} = \alpha_{ki}$ that

$$\frac{\nu_1}{E_1} = \frac{\nu_2}{E_2}.$$

For an isotropic homogeneous plate of constant rigidity, M_1, M_2 and H, as is known, have the following form:

$$
\begin{aligned}
M_1 &= -D\left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right] \\
M_2 &= -D\left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right] \\
H &= -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y},
\end{aligned} \tag{9}
$$

where

$$D = \frac{EJ}{1-\nu^2}$$

is the cylindrical rigidity of the plate.

The condition of equilibrium of the plate in the second state, obviously, will have the following form:

$$
\begin{aligned}
A_2 = \iint \Big\{ & T_1\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + T_2\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + \\
& + S\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial^2 w}{\partial x\,\partial y}\right]\Big\}\,dx\,dy - \frac{1}{2}\iint\left\{M_1\frac{\partial^2 w}{\partial x^2} + M_2\frac{\partial^2 w}{\partial y^2} + 2H\frac{\partial^2 w}{\partial x\,\partial y}\right\}dx\,dy,
\end{aligned} \tag{10}
$$

where A_2 is the work of the external forces in the second state.

In the theory of buckling, we are interested in the equation which corresponds to the moment of transition of the plate from the first state to the second. This transition of the first state to the second evidently can be accomplished at the moment that the forces acting in the plane of the plate have exceeded a certain limit, albeit by a small quantity.

We shall call the system of forces that corresponds to this limit the critical system, and we shall call the state of the plate that corresponds to this system of forces the critical state.

At the moment of the critical state (at the boundary of stability) the plate can assume both a plane and a deflected shape, i.e., both the first and the second state of the plate is equally probable. At that moment, the work of the external forces of both the first and the second state will be equal to each other [17, 19]. All terms that contain u and v in this case will cancel out, and we shall obtain the sought Brian-Timoshenko equation:

$$-\frac{1}{2}\iint\left\{M_1\frac{\partial^2 w}{\partial x^2}+M_2\frac{\partial^2 w}{\partial y^2}+2H\frac{\partial^2 w}{\partial x\,\partial y}\right\}dx\,dy+\iint\left\{\frac{1}{2}T_1\left(\frac{\partial w}{\partial x}\right)^2+\right.$$
$$\left.+\frac{1}{2}T_2\left(\frac{\partial w}{\partial y}\right)^2+S\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}dx\,dy=0, \tag{11}$$

or, replacing the values of M_1, M_2 and H in equation (11) with their expressions /143
(7), we finally obtain:

$$\frac{1}{2}\iint\left\{J(x,y)\left[\alpha_{11}\frac{\partial^2 w}{\partial x^2}+\alpha_{12}\frac{\partial^2 w}{\partial y^2}+2\alpha_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right]\frac{\partial^2 w}{\partial x^2}+\right.$$
$$+J(x,y)\left[\alpha_{21}\frac{\partial^2 w}{\partial x^2}+\alpha_{22}\frac{\partial^2 w}{\partial y^2}+2\alpha_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right]\frac{\partial^2 w}{\partial y^2}+ \tag{12}$$
$$\left.+2J(x,y)\left[\alpha_{61}\frac{\partial^2 w}{\partial x^2}+\alpha_{62}\frac{\partial^2 w}{\partial y^2}+2\alpha_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right]\frac{\partial^2 w}{\partial x\,\partial y}\right\}dx\,dy+$$
$$+\iint\left\{\frac{1}{2}T_1\left(\frac{\partial w}{\partial x}\right)^2+\frac{1}{2}T_2\left(\frac{\partial w}{\partial y}\right)^2+S\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right\}dx\,dy=0.$$

If, however, we replace the values of M_1, M_2 and H in equation (11) by their common expressions (9), we shall obtain the well-known Brian-Timoshenko equation [16].

3. Stable, Unstable, and Neutral Equilibrium

Both equation (11) and (12) are the only necessary condition of equilibrium. The necessary and sufficient condition of an elastic system whose forces have a potential, as we know, is that the potential energy in the equilibrium position have a stationary value, i.e.,

$$\delta\Pi=0, \tag{13}$$

or, expressed in another manner,

$$\frac{\partial\Pi}{\partial q_i}=0 \qquad (i=1,2,3,\ldots,n), \tag{14}$$

where Π is the total potential energy of the system and q_i is a generalized coordinate.

The equilibrium of a material system, as we know, can be stable, unstable, or neutral. If the potential energy has a minimum, i.e.,

$$\delta^2\Pi>0, \tag{15}$$

the equilibrium will be stable.

It is has a maximum, i.e.,

$$\delta^2\Pi < 0, \tag{16}$$

the equilibrium will be unstable; and if it has neither a minimum nor a maximum, i.e.,

$$\delta^2\Pi = 0 \text{ and } \delta\Pi = 0, \tag{17}$$

the equilibrium will be neutral.

Neutral equilibrium determined by equations (17) will obviously correspond to the critical state of equilibrium.

4. Expression of the Total Potential Energy of a Plate /144

To find the critical state, it is obviously necessary to derive an expression for the total potential energy. In our case, the left-hand side of equation (12) may be considered as the total potential energy of a plate subject to buckling.

Indeed, the first double integral in equation (12) is none other than the work of the internal bending forces A_i, while the second, if it is read with a minus sign, is the work of the external forces A_e (boundary and volume forces). Consequently, the total potential energy for the case of buckling can be written in the following form:

$$\Pi = A_e - A_i \tag{18}$$

or

$$\begin{aligned}\Pi = \frac{1}{2}\iint J(x, y)\Big\{&\Big[\alpha_{11}\frac{\partial^2 w}{\partial x^2} + \alpha_{21}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{61}\frac{\partial^2 w}{\partial x\,\partial y}\Big]\frac{\partial^2 w}{\partial x^2} + \\ &+\Big[\alpha_{21}\frac{\partial^2 w}{\partial x^2} + \alpha_{22}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{62}\frac{\partial^2 w}{\partial x\,\partial y}\Big]\frac{\partial^2 w}{\partial y^2} + \\ &+2\Big[\alpha_{61}\frac{\partial^2 w}{\partial x^2} + \alpha_{62}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{66}\frac{\partial^2 w}{\partial x\,\partial y}\Big]\frac{\partial^2 w}{\partial x\,\partial y}\Big\}\,dx\,dy + \\ &+\iint\Big\{\frac{1}{2}T_1\Big(\frac{\partial w}{\partial x}\Big)^2 + \frac{1}{2}T_2\Big(\frac{\partial w}{\partial y}\Big)^2 + S\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\Big\}\,dx\,dy.\end{aligned} \tag{19}$$

5. Differential Equation of the Elastic Surface of a Plate and Boundary Conditions

We shall use condition (13) to obtain the differential equation and the boundary conditions.

By inserting the value of Π from (19) into (13) and performing variation and some transformations, we obtain:

$$\iint dx\,dy \left\{ \frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 M_2}{\partial y^2} + \frac{\partial}{\partial x}\left[T_1\frac{\partial w}{\partial x} + S\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial y}\left[T_2\frac{\partial w}{\partial y} + S\frac{\partial w}{\partial x}\right]\right\}\delta w - \tag{20}$$
$$- \int ds \left\{ Q\delta w - M\frac{\partial \delta w}{\partial n}\right\} - \sum_{i=1} S_i(N)\,\delta w(s_i),$$

where

$$Q = \left\{ \left(\frac{\partial M_1}{\partial x} + \frac{\partial H}{\partial y}\right)\cos(n,x) + \left(\frac{\partial M_2}{\partial y} + \frac{\partial H}{\partial x}\right)\cos(n,y) - \right.$$
$$- \frac{\partial}{\partial s}\left[(M_1 - M_2)\cos(n,x)\cos(n,y) - H\{\cos^2(n,x) - \cos^2(n,y)\}\right] + \tag{21}$$
$$\left. + \left[T_1\frac{\partial w}{\partial x} + S\frac{\partial w}{\partial y}\right]\cos(n,x) + \left[T_2\frac{\partial w}{\partial y} + S\frac{\partial w}{\partial x}\right]\cos(n,y)\right\};$$

$$M = M_1\cos^2(n,x) + 2H\cos(n,x)\cos(n,y) + M_2\cos^2(n,y), \tag{22}$$
$$N = (M_1 - M_2)\cos(n,x)\cos(n,y) - H[\cos^2(n,x) - \cos^2(n,y)],$$

in which $\sum_{i=1} S_i(N)$ denotes the sum of the discontinuities which the function N 145
undergoes at the corner points s_i of the contour (n,x), and (n,y) are the angles between the external direction of the normal to the contour and the x and y axes; the quantities M_1, M_2 and H have their previous values (7).

From (20), with the usual considerations, we obtain an Euler-Lagrange equation of our variation problem:

$$\frac{\partial^2 M_1}{\partial x^2} + 2\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 M_2}{\partial y^2} + \frac{\partial}{\partial x}\left[T_1\frac{\partial w}{\partial x} + S\frac{\partial w}{\partial y}\right] + \frac{\partial}{\partial y}\left[T_2\frac{\partial w}{\partial y} + S\frac{\partial w}{\partial x}\right] = 0, \tag{23}$$

or, by substituting the values of M_1, M_2 and H in (23) by their expressions (7),

$$\frac{\partial^2}{\partial x^2}\left\{J(x,y)\left[\alpha_{11}\frac{\partial^2 w}{\partial x^2} + \alpha_{12}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right]\right\} +$$
$$+ 2\frac{\partial^2}{\partial x\,\partial y}\left\{J(x,y)\left[\alpha_{61}\frac{\partial^2 w}{\partial x^2} + \alpha_{62}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right]\right\} + \tag{24}$$
$$+ \frac{\partial^2}{\partial y^2}\left\{J(x,y)\left[\alpha_{21}\frac{\partial^2 w}{\partial x^2} + \alpha_{22}\frac{\partial^2 w}{\partial y^2} + 2\alpha_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right]\right\} =$$
$$= T_1\frac{\partial^2 w}{\partial x^2} + 2S\frac{\partial^2 w}{\partial x\,\partial y} + T_2\frac{\partial^2 w}{\partial y^2} - X\frac{\partial w}{\partial x} - Y\frac{\partial w}{\partial y},$$

where X and Y are the components of the resultant vector of the volume forces applied to a unit area of the plate.

Equation (23) also can be obtained geometrically after considering the equilibrium of a deformed element of the plate.

The static equation $\sum Z = 0$ also gives the sought equation (23)*.

The equation obtained (24) makes it possible to write differential equations also for plates with other values of $\alpha_{ik}(x,y)$ and $J(x,y)$.

We shall cite several particular cases of differential equation (24).

1. To obtain a differential equation for an orthogonal anisotropic plate of variable thickness, we must place the following in equation (24):

$$\alpha_{16}=\alpha_{61}=\alpha_{26}+\alpha_{62},$$

which is equivalent to

$$\beta_1=\beta_2=0,$$

The remaining coefficients will then take on the following form:

$$\alpha_{11}=\frac{E_1}{1-\nu_1\nu_2},\quad \alpha_{12}=\frac{\nu_2 E_1}{1-\nu_1\nu_2},\quad \alpha_{66}=G,$$

$$\alpha_{21}=\frac{\nu_1 E_2}{1-\nu_1\nu_2},\quad \alpha_{22}=\frac{E_2}{1-\nu_1\nu_2},\quad K=\frac{1-\nu_1\nu_2}{E_1E_2G}.$$

2. Equation (24) for the case of a homogeneous orthogonal anisotropic plate of constant thickness, takes on the form: /146

$$A\frac{\partial^4 w}{\partial x^4}+2C\frac{\partial^4 w}{\partial x^2\partial y^2}+B\frac{\partial^4 w}{\partial y^4}=\frac{\partial}{\partial x}\left[T_1\frac{\partial w}{\partial x}+S\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial y}\left[T_2\frac{\partial w}{\partial y}+S\frac{\partial w}{\partial x}\right],$$

where

$$A=\frac{E_1J}{1-\nu_1\nu_2},$$

$$B=\frac{E_2J}{1-\nu_1\nu_2},$$

$$2C=\frac{\nu_1E_2J}{1-\nu_1\nu_2}+\frac{\nu_2E_1J}{1-\nu_1\nu_2}+4GJ.$$

---

*Equation (23), and also expressions (21) and (22), obviously are invariants both with respect to the structure of the plate material and also with regard to the variation of its rigidity.

3. In an isotropic and homogeneous medium, but with variable thickness, equation (24) assumes the following form:

$$\Delta(D\Delta w)-(1-\nu)\left[\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\,\partial y}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right]=$$
$$=\frac{\partial}{\partial x}\left[T_1\frac{\partial w}{\partial a}+S\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial y}\left[T_2\frac{\partial w}{\partial y}+S\frac{\partial w}{\partial x}\right], \tag{25}$$

where $D = EJ/1 - \nu^2$ is the cylindrical rigidity of the plate.

4. When D = const, equation (25) becomes a differential equation for an isotropic plate with constant thickness:

$$D\Delta\Delta w=\frac{\partial}{\partial x}\left[T_1\frac{\partial w}{\partial x}+S\frac{\partial w}{\partial y}\right]+\frac{\partial}{\partial y}\left[T_2\frac{\partial w}{\partial y}+S\frac{\partial w}{\partial x}\right]; \tag{26}$$

this equation was first derived by Reissner [15]. If we state the following in equation (26),

$$T_1=\text{const},\quad T_2=\text{const and } S=\text{const},$$

we will obtain a Saint-Venant equation [20].

We shall obtain the boundary conditions for the plate by considering the contour integral and the sum in the equation (20). If the edge of the plate is perfectly free, then, by using the well-known reasoning, we obtain the following:

$$Q=0,\qquad M=0,\qquad \sum_{i=1}^{\infty}S_i(N)=0, \tag{27}$$

where Q, M and N have their former values (21), (22), (22a).

Thus, for example, for the free side $y = b$ of a rectangular plate, the boundary conditions (27) assume the following form: /147

$$\left[\frac{\partial}{\partial y}\left\{J(x,y)\left(\alpha_{21}\frac{\partial^2 w}{\partial x^2}+\alpha_{22}\frac{\partial^2 w}{\partial y^2}+2\alpha_{62}\frac{\partial^2 w}{\partial x\,\partial y}\right)\right\}+\right.$$
$$\left.+\frac{\partial}{\partial x}\left\{J(x,y)\left(\alpha_{61}\frac{\partial^2 w}{\partial x^2}+\alpha_{62}\frac{\partial^2 w}{\partial y^2}+2\alpha_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right)\right\}-T_2\frac{\partial w}{\partial y}-S\frac{\partial w}{\partial x}\right]_{y=b}=0, \tag{28}$$
$$\left[\alpha_{21}\frac{\partial^2 w}{\partial x^2}+\alpha_{22}\frac{\partial^2 w}{\partial y^2}+2\alpha_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right]_{y=b}=0;$$

$x = 0$, $y = b$ and $x = a$, $y = b$ at the corners of the plate:

$$S_i\left(-J(x,y)\left[\alpha_{61}\frac{\partial^2 w}{\partial x^2}+\alpha_{62}\frac{\partial^2 w}{\partial y^2}+2\alpha_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right]\right)\delta w(s_i)=0. \tag{28'}$$

The boundary conditions for an isotropic homogeneous plate of constant rigidity will be obtained if, instead of M_1, M_2 and H in (27), we substitute their common expressions (9). In particular, for the free side $y = b$ of a rectangular plate, the boundary conditions take on the following form:

$$\left[D\left(\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x^2\partial y}\right)-T_2\frac{\partial w}{\partial y}-S\frac{\partial w}{\partial x}\right]_{y=b}=0,$$
$$\left[\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right]_{y=b}=0; \tag{28a}$$

$x = 0$, $y = b$ and $x = a$, $y = b$ at the corners of the plate:

$$S_i\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)\delta w(s_i)=0. \tag{28a'}$$

The usual Kirchhoff boundary conditions encountered in all courses on the theory of elasticity will be obtained if we place $(T_2)_{y=b} = (S)_{y=b} = 0$ in (28a), which corresponds to the case of the absence of external forces on the contour. But since cases are encountered in the literature where Kirchhoff's boundary conditions are extended to individual particular problems and to loaded sides of a plate, which naturally leads to errors, we then must consider this question in greater detail and cite an example.

As an example, let us consider the stability of the rod shown in Figure 1 in two variations: with the boundary conditions given in this article, and with Kirchhoff's boundary conditions.

The differential equation of the elastic line of the rod will be obtained as a particular case from (24):

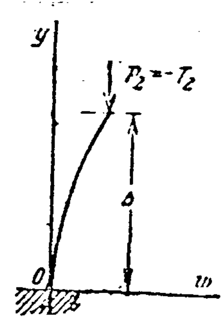

Figure 1.

$$-\frac{d^2}{dy^2}\left(EJ\frac{d^2 w}{dy^2}\right)+T_2\frac{d^2 w}{dy^2}=0. \tag{29}$$

The boundary conditions for the free loaded end of the rod will be obtained as a particular case from (28a): /14

$$\left[-EJ\frac{d^3 w}{dy^3}+T_2\frac{dw}{dy}\right]_{y=b}=0, \qquad \left[\frac{d^2 w}{dy^2}\right]_{y=b}=0. \tag{30}$$

Kirchhoff's boundary conditions for the same free end of the rod have the following form:

$$\left[\frac{d^3w}{dy^3}\right]_{y=b}=0, \qquad \left[\frac{d^2w}{dy^2}\right]_{y=b}=0. \tag{31}$$

The boundary conditions for the fastened end, which are given here, and Kirchhoff's boundary conditions, coincide:

$$[w]_{y=b}=0, \qquad \left[\frac{dw}{dy}\right]_{y=b}=0. \tag{32}$$

Now, solving the problem for a rod under boundary conditions (30) and (32) and boundary conditions (31) and (32), we obtain the following transcendental equation for determining the critical force in the first case:

$$\cos\lambda b=0 \qquad \left(\lambda=\sqrt{\frac{P_2}{EJ}}\right) \tag{33}$$

and the following transcendental equation for the second case:

$$\sin^2\lambda b+\cos^2\lambda b=0. \tag{33'}$$

Equation (33'), as we know, does not generally have a root, and consequently, we do not obtain the critical force*.

This example indicates that Kirchhoff's boundary conditions, which were derived for bending, cannot always be extended to buckling.

Let us now consider other cases of boundary conditions.

If the edge of a plate is freely fastened, δw = 0 and $\delta w(s_i)$ = 0, and consequently, by applying well-known reasoning, we obtain the following:

$$w=0, \quad M_1\cos^2(n,x)+2H\cos(n,x)\cos(n,y)+M_2\cos^2(n,y)=0. \tag{34}$$

If the edge of the plate is rigidly fastened, i.e., it cannot be turned or displaced, the boundary conditions, as can be easily comprehended, will have the following form:

$$w=0, \qquad \frac{\partial w}{\partial n}=0. \tag{35}$$

*A similar example could be cited also for a plate, but the incorrectness of applying Kirchhoff's conditions in that case is not as clear to detect as in this case.

The boundary conditions could be derived also for the case of elastic fastening of the edge of a plate, and also for the case of support of the edge of a plate on an elastic contour, but in the presence of the fundamental boundary conditions given above, a description of these boundary conditions is not difficult; therefore, we shall not dwell on them. /149

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